

On the uniform distribution of the Prüfer angles and its implication for a sharp spectral transition of Jacobi matrices with randomly sparse perturbations

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Abstract

In the present work we consider Jacobi matrices with random uncertainty in the position of sparse perturbations. We prove (Theorem 3.2) that the sequence of Prüfer angles $(\theta_k^\omega)_{k \geq 1}$ is u.d mod π for all $\varphi \in [0, \pi]$ with exception of a set of rational multiples of π and for almost every ω with respect to the product $\nu = \prod_{j \geq 1} \nu_j$ of uniform measures on $\{-j, \dots, j\}$. Together with an improved criterion for pure point spectrum (Lemma 4.1), this provides a simple and natural alternative proof of a result of Zlatos (J. Funct. Anal. **207**, 216-252 (2004)): the existence of pure point (p.p) spectrum and singular continuous (s.c.) spectra on sets complementary to one another with respect to the essential spectrum $[-2, 2]$, outside sets A_{sc} and A_{pp} , respectively, both of zero Lebesgue measure (Theorem 2.4). Our method allows for an explicit characterization of A_{pp} , which is seen to be also of dense p.p. type, and thus the spectrum is proved to be exclusively pure point on one subset of the essential spectrum.

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1 Introduction

In [Z], Zlatoš proved, among several others, a result asserting that, for a class of sparse random Schrödinger operators $H_\phi^{(\omega)}$ on \mathbb{Z}_+ , equivalent to the Jacobi matrices $J_{P,\phi}$ defined below, its spectral measure $\mu_\phi^{(\omega)}(\lambda)$, restricted to the interval

$$K = \left(-\sqrt{4 - v^2/(\beta - 1)}, \sqrt{4 - v^2/(\beta - 1)} \right), \quad v^2 < 4(\beta - 1) \quad (1.1)$$

for a.e. disorder ω ($\omega = \{\omega_n, n \geq 1\}$ are independent random variables uniformly distributed over $\{-n, -n + 1, \dots, n\}$) and a.e. phase boundary ϕ , is purely singular continuous (s.c.) with local Hausdorff dimension

$$1 - \frac{\ln \left(1 + \frac{v^2}{4 - \lambda^2} \right)}{\ln \beta} \quad (1.2)$$

and dense pure point (p.p.) in the rest of the interval $[-2, 2]$. Although both the existence of p.p. spectrum and the sharpness of the transition were mentioned without explicit proof (Theorem 6.3 of [Z]), such a proof exists [R]. Theorem 6.3 is preceded by a two-page argument establishing that the following statements hold for almost ϕ and for almost all points ω in the probability space: for almost all energies $\lambda \in (-2, 2)$ there is a subordinate generalised eigenfunction with $O(n^{-\alpha_\lambda})$ decay and all other generalised eigenfunctions have n^{α_λ} growth; $\alpha_\lambda = \log(1 + v^2/(4 - \lambda^2)) / (2 \log \beta)$, and thus, for $\alpha_\lambda \leq 1/2$, i.e., the forthcoming interval I of Theorem 2.4, assertion (a) there holds, while in the complementary set¹ assertion (b) there is valid (but without the explicit characterization (2.26)).

The excluded set A of Lebesgue measure zero in $(-2, 2)$, mentioned in the above Theorem (A has been written as the union of the two sets A_{sc} and A_{pp} situated in disjoint regions), is a common feature of both methods of [Z] and ours, and is due to the peculiarities of the definition of the essential support or minimal support of a measure (see Definition 1 of [GP]). Such set are not present in the cases of purely p.p. or purely s.c. spectrum (see Theorems 5.1 and 5.2) but occurs here. The set A_{sc} might be a discrete set or even dense p.p., the set A_{pp} a countable union of a collection of (eventually different) Cantor sets of zero Lebesgue measure, in which case both spectra would be mixed. One distinctive feature of our method is that it allows us to make A_{pp} explicit, which is seen also to be dense p.p., so that at least the p.p. part of the spectrum is not mixed (see (2.26)). We believe the same happens for the s.c. part, but are not able to prove it.

Our objective is to prove both the existence of p.p. spectrum and the sharpness of the transition with (s.c.) spectrum (Theorem 2.4) by a novel method, whose investigation started in [MWGA] and continued in [CMW1]. The existence proof involves the ergodicity of the Prüfer angles, corresponding to a solution u of the eigenvalue equation $J_{P,\phi}u = \lambda u$. As remarked by C. Remling [Re] in his review of our Nonlinearity paper [MWGA], which introduced our method, our new idea was

¹For $\alpha_E > 1$ the subordinate generalized eigenfunction is an $l_2(\mathbb{Z}_+)$ eigenfunction.

to **fix** the energy and assume the Prüfer angles $(\theta_j^\omega)_{j \geq 1}$ at a_j are uniformly distributed (u.d.) as a function of j – instead of the traditional approach which exploits the u.d. of the Prüfer angles in the energy variable at fixed a_j . In [MWGA] we were only able, however, to fix the energy in the s.c. spectrum. In the present paper we are able to bring our ideas to full fruition: we prove that the Prüfer angles are indeed u.d. (in the above sense) for any fixed energy, be it in the s.c. or p.p. spectrum, by exploiting a (slight) modification of the (optimal) metric extension of Weyl’s criterion for uniform distribution by Davenport, Erdős and LeVeque [DEL] (see also [KN], Theorem 4.2) – this is our Theorem 3.2. Due to its simplicity and naturalness, we expect that the method will find a wider range of applications.

Our version of the sharpness of the spectral transition in Theorem 6.3 of [Z] proceeds along lines different from [Z]. Section 4, devoted to the issue, replaces (2.20) by a criterion for pure point spectrum (Lemma 4.1) that is suitable for sparse potentials. Lemma 4.1 is the analogue of Lemma 2.1 of [Z] in which the decay of a subordinate solution is obtained.

It is worth noting that a sharp spectral transition occurs because both the sparsity and the norm of transfer matrices grow exponentially with respect to j . This conclusion is confirmed for models whose sparseness grow with sub or super-exponential rates. For sub-exponential sparseness, the spectrum is dense pure point (Theorem 5.1); for super-exponential sparseness, the spectrum is purely singular continuous (Theorem 5.2). Varying the intensity of perturbation, a model with super-exponential sparseness may have purely singular continuous spectrum of Hausdorff dimension 0 (Theorem 5.4). It is, nevertheless, possible to choose $\delta = 1$ in relation (5.7) in order to obtain a (possibly not sharp) transition from singular continuous to pure point spectrum with the spectral measure being of Hausdorff dimension 0 (see Remark 5.6).

We emphasize that no familiarity of the reader with [MWGA] is assumed. He (she) may look at Section 2 for notation, Theorem 2.4 for the statement of the main result and then concentrate on Theorems 3.1 and 3.2, Lemma 4.1 and the completion of the proof of Theorem 2.4 after Lemma 4.1, which sketches all basic steps of the proof. He (she) will turn eventually to pages 776–778 of [MWGA] to fill in some details of the derivation, which follows the method of [LS] and will not be repeated there.

The present paper is organized as follows. In Section 2 we present the model, some definitions and results regarding its spectral transition. In Section 3 we prove Theorem 3.2 on the Prüfer angles. In Section 4 we improve the criterion for pure point spectrum. Section 5 contains Theorems 5.1, 5.2 and 5.4 which discuss subexponential and superexponential sparsity. In Section 6 we briefly relate the present result to a program started by Molchanov [Mo] concerning an Anderson-like transition.

As a last remark, the Hausdorff dimension of the spectral measure is readily obtained by applying the Gilbert–Pearson subordinacy criterion [GP] adapted for sparse potentials by Jitomirskaya and Last: see [Z] for the diagonal case, [CMW1] for the nondiagonal case, by our method. We refer to [Re] for additional references.

2 Some Facts Regarding the Studied Class of Jacobi Matrices

We begin with some definitions and basic results. The model analyzed here is a small variation of the model studied in [MWGA], i.e., a class of nondiagonal Jacobi matrices subject to exponentially sparse perturbations of finite size. Although we consider this model for definiteness, because it is part of a long-term project started in [MWGA], and further developed in [CMW1], we emphasize that the result apply with a few minor changes to the diagonal model considered by Zlatoš [Z] – it is enough to replace $(1 - p^2)/p$ in expression (2.23) by the intensity parameter v of equations (1.1) and (1.2).

We consider off-diagonal Jacobi matrices

$$J_P = \begin{pmatrix} 0 & p_0 & 0 & 0 & \cdots \\ p_0 & 0 & p_1 & 0 & \cdots \\ 0 & p_1 & 0 & p_2 & \cdots \\ 0 & 0 & p_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (2.1)$$

for each sequence $P = (p_n)_{n \geq 0}$ of the form

$$p_n = \begin{cases} p & \text{if } n = a_j^\omega \in \mathcal{A}, \\ 1 & \text{if otherwise,} \end{cases} \quad (2.2)$$

for $p \in (0, 1)$. The novelty with respect to [MWGA]: $\mathcal{A} = \{a_j^\omega\}_{j \geq 1}$ is now a random set of natural numbers $a_j^\omega = a_j + \omega_j$ with a_j satisfying the "sparseness" condition

$$a_j - a_{j-1} = \beta^j, \quad j = 2, 3, \dots \quad (2.3)$$

with $a_1 + 1 = \beta \geq 2$ and ω_j , $j \geq 1$, independent random variables defined on a probability space $(\Omega, \mathcal{B}, \mu)$, uniformly distributed on the set $\Lambda_j = \{-j, \dots, j\}$. These variables introduce uncertainty in the position of the points where the p_n differs from one. Such models are nowadays called Poisson models, see [SJ] and references given there. Note that the support of ω_j increases only linearly with respect to the index j .

The Jacobi matrices (2.1) can be written as

$$(J_P u)_n = p_n u_{n+1} + p_{n-1} u_{n-1},$$

with $u = (u_n)_{n \geq 0} \in l_2(\mathbb{Z}_+)$ and we denote by $J_{P,\phi}$ the Jacobi matrix J_P which satisfies a ϕ -boundary condition at -1 :

$$u_{-1} \cos \phi - u_0 \sin \phi = 0. \quad (2.4)$$

Remark 2.1 $J_{P,\phi}$ is a (noncompact) perturbation of the free Jacobi matrix $J_{1,\phi}$, where $p_n = 1$ for all $n \geq 0$: $J_{P,\phi} = J_{1,\phi} + V_P$, the "potential" V_P composed by infinitely many random barriers whose distances grow exponentially fast. A disordered potential of this sort was introduced by Zlatoš [Z].

2.1 Essential spectrum

Since, for some fixed realization of ω , we are dealing with an operator $J_{P,\phi}$ already studied on [MWGA], its essential spectrum (for a definition, see section VII.3 of [RS]) remains the same with probability 1:

Theorem 2.2 *The essential spectrum of the Jacobi matrix $J_{P,\phi}$, defined by (2.1), (2.2) and satisfying the ϕ -boundary condition (2.4), for each realization of $\omega_j \in \Lambda_j \equiv \{-j, \dots, j\}$, $j \geq 1$, is given by*

$$\sigma_{ess} = [-2, 2] . \quad (2.5)$$

The proof of Theorem 2.2 is as the proof of Theorem 2.1 of [MWGA].

2.2 Prüfer variables

To fix notation, we shall briefly review Sections 3 and 4 of [MWGA] incorporating eventual adjustments due to the randomness on the position of the perturbations.

For $\lambda \in \mathbb{C}$, let

$$T(n, n-1; \lambda) = \begin{pmatrix} \frac{\lambda}{p_n} & \frac{-p_{n-1}}{p_n} \\ 1 & 0 \end{pmatrix} \quad (2.6)$$

be the 2×2 transfer matrix associated to the $l_2(\mathbb{Z}_+)$ solution of the eigenvalue equation $J_{P,\phi}u = \lambda u$. The equation

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = T(n, n-1; \lambda) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix} \quad (2.7)$$

holds for $n \geq 0$, with $\begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix} = \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}$ satisfying (2.4) for some $\phi \in [0, \pi]$. The product of the $n+1$ first transfer matrices is denoted by

$$T(n; \lambda) = T(n, n-1; \lambda)T(n-1, n-2; \lambda) \dots T(0, -1; \lambda) . \quad (2.8)$$

Given the definition (2.2) of p_n and the sparseness condition (2.3), only three different 2×2 matrices appear in the r.h.s. of (2.8):

$$T_-(\lambda) = \begin{pmatrix} \lambda/p & -1/p \\ 1 & 0 \end{pmatrix} , \quad T_+(\lambda) = \begin{pmatrix} \lambda & -p \\ 1 & 0 \end{pmatrix} \text{ or } \quad T_0(\lambda) = \begin{pmatrix} \lambda & -1 \\ 1 & 0 \end{pmatrix} \quad (2.9)$$

occurs depending on whether the left, the right or none of the two entries n and $n-1$ in (2.6) are equal to $a_j^\omega \in \mathcal{A}$. The free matrix $T_0(\lambda)$ is similar to a pure clockwise rotation by $\varphi \in (0, \pi)$:

$$UT_0(\lambda)U^{-1} = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} = R(\varphi) , \quad (2.10)$$

where $\lambda = 2 \cos \varphi$ and the similarity matrix is

$$U \equiv \begin{pmatrix} 0 & \sin \varphi \\ 1 & -\cos \varphi \end{pmatrix}. \quad (2.11)$$

Note that U is not uniquely defined since any other matrix $U' = HU$, with H commuting with R , satisfies (2.10).

For every $n \in \mathbb{N}$ and $\omega_j \in \{-j, \dots, j\}$, $j \geq 1$, there is an integer k such that $a_k^\omega \leq n < a_{k+1}^\omega$ and the conjugation of (2.8) by U^{-1} ,

$$UT(n; \lambda)U^{-1} = R((n - a_k^\omega)\varphi)P_{+-}(\lambda)R((a_k^\omega - a_{k-1}^\omega)\varphi) \cdots P_{+-}(\lambda)R((a_1^\omega)\varphi), \quad (2.12)$$

intertwines P_{+-} defined by

$$R(\varphi)P_{+-}(\lambda)R(\varphi) = UT_+(\lambda)T_-(\lambda)U^{-1}$$

with rotations.

Finally, (2.12) are used together with (2.7) and (2.8) to define the Prüfer variables $(R_k, \theta_k^\omega)_{k \geq 0}$. Observe that the Prüfer angles are now random variables. The Prüfer radius, on the other hand, are random variables only as a function of the Prüfer angles and their dependence on ω will be omitted. Like in Section 3 of [MWGA] (more specifically, equations (3.3)–(3.7)), if

$$v_k = (R_{k-1} \cos \theta_k^\omega, R_{k-1} \sin \theta_k^\omega)$$

$k = 1, 2, \dots$, are defined by

$$v_k = R((a_k^\omega - a_{k-1}^\omega)\varphi)P_{+-}(\lambda)v_{k-1}$$

for $k > 1$ with $v_1 = R(a_1^\omega\varphi)v_0$ where

$$P_{+-}(\lambda) = \begin{pmatrix} p & 0 \\ \frac{1-p^2}{p} \cot \varphi & \frac{1}{p} \end{pmatrix}, \quad (2.13)$$

then the Prüfer variables satisfy a recursive relation (see equations (2.15)–(2.17) below).

By equivalence of norms, the growth of $T(n; \lambda)$ may be controlled by the euclidean norm (see Section 4 of [MWGA] for detail):

$$\|UT(n; \lambda)U^{-1}v_0\|^2 = \|UT(a_N^\omega + 1; \lambda)U^{-1}v_0\|^2 = R_N^2, \quad (2.14)$$

where the equality holds for any unit vector $v_0 = (\cos \theta_0, \sin \theta_0)$ and for each n such that $a_N^\omega \leq n < a_{N+1}^\omega$. Thus, from equations (2.10)–(2.13), R_N^2 can be written as

$$(R_N^2)^{1/N} = \prod_{j=1}^N \left(\frac{R_j^2}{R_{j-1}^2} \right)^{1/N} = \frac{1}{p^2} \exp \left\{ \frac{1}{N} \sum_{k=1}^N f(\theta_k^\omega) \right\}, \quad (2.15)$$

where

$$f(\theta) = \ln(p^4 \cos^2 \theta + (\sin \theta + (1 - p^2) \cot \varphi \cos \theta)^2) \quad (2.16)$$

and the Prüfer angles $(\theta_k^\omega)_{k \geq 1}$ are defined recursively by

$$\theta_k^\omega = \tan^{-1} \left(\frac{1}{p^2} (\tan \theta_{k-1}^\omega + \cot \varphi) - \cot \varphi \right) - (\beta^k + \omega_k - \omega_{k-1})\varphi \quad (2.17)$$

for $k > 1$ with θ_1^ω given by

$$\theta_1^\omega = \theta_0 - (a_1 + \omega_1)\varphi.$$

2.3 Existence of singular continuous spectrum

The nature of the spectrum of $J_{P,\phi}$ is intimately related to the growth of the transfer matrix $T(n; \lambda)$. This connection was precisely stated by Last and Simon [LS]: the essential support Σ_{ac} of the absolutely continuous part μ_{ac} of the spectral measure $\mu = d\rho$ is given by

$$\Sigma_{\text{ac}} = \left\{ \lambda : \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \|T(n; \lambda)\|^2 < \infty \right\}. \quad (2.18)$$

By Theorem 1.6 of [LS], a sufficient condition for $\lambda \in \sigma_{\text{sc}}$, the singular-continuous spectrum, is that

$$\sum_{n=0}^{\infty} \|T(n; \lambda)\|^{-2} = \infty, \quad (2.19)$$

since the eigenvalue equation $J_{P,\phi} u = \lambda u$ has no $l_2(\mathbb{Z}_+)$ -solutions. Theorem 1.7 of [LS] asserts that the eigenvalue equation has an $l_2(\mathbb{Z}_+)$ solution (which is precisely a sufficient condition for $\lambda \in \sigma_{\text{pp}}$, the pure-point spectrum) if

$$N_{\text{pp}} \equiv \sum_{n=0}^{\infty} \|T(n; \lambda)\|^2 \left(\sum_{k=n}^{\infty} \|T(k; \lambda)\|^{-2} \right)^2 < \infty. \quad (2.20)$$

These relations were employed by Marchetti et. al. in [MWGA] to determine the spectral nature of the Jacobi matrices (with sparse perturbations in deterministic positions). As we shall see, condition (2.20) can be improved if Theorem 8.1 of [LS], used to establish (2.20), is applied for the subsequence $T(a_k^\omega + 1; \lambda)$, $k \geq 1$, instead of $T(n; \lambda)$ for any $n \in \mathbb{Z}_+$ (see Section 4). This improvement removes the gap in the spectrum between the two intervals (2.24) and (2.25).

Provided the Prüfer angles $(\theta_k^\omega)_{k \geq 1}$ defined by (2.17) are uniformly distributed modulo π , the Birkoff sum in the r.h.s of (2.15) can be replaced, as N tends to infinity, by an integral over Lebesgue measure $d\theta$. Thus, the simple knowledge of the integral of $f(\theta)$ gives us the asymptotic behavior of R_N . Evidently, the assumption for this replacement has to be proven.

To formulate the questions addressed and enunciate our results, let us state precisely what has been proved in [MWGA]. We say that the Prüfer angles $(\theta_k)_{k \geq 1}$ have uniform distribution modulo π (u.d. mod π) if

$$\lim_{N \rightarrow \infty} \frac{\text{card}(\{k : \theta_k \bmod \pi \in [0, \theta), 1 \leq k \leq N\})}{N} = \theta ,$$

holds for any $\theta \in [0, \pi]$, where $\text{card}(S)$ is the number of elements in S . By the ergodic theorem (see e.g. Theorem 1.1 of [KN]), the sequence $(\theta_k)_{k \geq 1}$ of real numbers is u.d. mod π if, and only if,

$$\lim_{n \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N f(\theta_k) = \frac{1}{\pi} \int_0^\pi f(\theta) d\theta \quad (2.21)$$

holds for every real-valued periodic Riemann integrable function f of period π .

It is easy to prove that f defined by (2.16) satisfies all those requirements and its integral is given by

$$\frac{1}{\pi} \int_0^\pi f(\theta) d\theta = \ln(rp^2) \quad (2.22)$$

where

$$r = 1 + \frac{(1 - p^2)^2}{4p^2} \text{cosec}^2 \varphi . \quad (2.23)$$

(see e.g. equations (4.14)-(4.17) of [MWGA]). Hence, the ergodic theorem (2.21) combined with equation (2.15) and relations (2.18)-(2.20), lead to Theorem 4.3 of [MWGA], which we state for completeness:

Theorem 2.3 *Let $J_{P,\phi}$ be the Jacobi matrix given by (2.1), with the sequence $P = (p_n)_{n \geq 0}$ defined by relations (2.2) and (2.3) and which satisfies the boundary condition (2.4). Let*

$$I_1 \equiv \left\{ \lambda \in [-2, 2] : \frac{p^2}{(1 - p^2)^2} (\beta - 1)(4 - \lambda^2) \geq 1 \right\} \quad (2.24)$$

$$I_2 \equiv \left\{ \lambda \in [-2, 2] : \frac{p^2}{(1 - p^2)^2} (\beta^3 - 1)(4 - \lambda^2) < 1 \right\} \quad (2.25)$$

with $p \in (0, 1)$ and $\beta \in \mathbb{N}$, $\beta \geq 2$. If, for each realization of ω , the Prüfer angles $(\theta_k^\omega)_{k \geq 1}$ are u.d. mod π for $\varphi \in [0, \pi] \setminus A_{\theta_0}^\omega$, $A_{\theta_0}^\omega$ a set of zero Lebesgue measure, possibly θ_0, ω -dependent, then

(a) *there exists a set A_1 of Lebesgue measure zero such that the spectrum restricted to the set $I_1 \setminus A_1$ is purely singular continuous,*

(b) *the spectrum of $J_{P,\phi}$ is pure point when restricted to I_2 for almost every $\phi \in [0, \pi)$.*

The proof of theorem 2.3 is entirely contained in Section 4 of [MWGA].

For the deterministic model considered in [MWGA] – in particular, for the model defined here for fixed ω – only numerical evidence for the uniform distribution of the Prüfer angles $(\theta_k^\omega)_{k \geq 1}$ has been provided. The Prüfer angles $(\theta_k^\omega)_{k \geq 1}$ may, however, be replaced by an u.d. mod π sequence

$(\zeta_k)_{k \geq 1}$ where $\zeta_k(\varphi)$, as a function of φ , is a continuous piecewise linear interpolation of $\theta_k(\varphi)$. It is shown in [MWGA] that the error E of replacing their corresponding Birkoff sum can be made as small as one wishes, provided the sparseness parameter $\beta > \beta_0(E, \lambda, p)$ is large enough. The problem is that (E, λ, p) has to be fixed and $\beta > \beta_0$ makes the set I_2 given by (2.25) an empty set. Theorem 5.8 of [MWGA] states that only the singular continuous part of Theorem 2.3 is present for $\beta > \beta_0$ if the hypothesis on the Prüfer angles is removed.

2.4 Transition from singular continuous to dense pure point spectrum

In the present work we prove (Theorem 3.2) that the sequence of Prüfer angles $(\theta_k^\omega)_{k \geq 1}$ is u.d. mod π for all $\varphi \in [0, \pi]$ with exception of the set of rational multiples of π and for almost every ω with respect to the product $\nu = \prod_{j \geq 1} \nu_j$ of uniform measures on Λ_j . This, together with Theorem 2.3, establishes the existence of a spectral transition from singular continuous to dense pure point spectrum. We also prove in Section 4 a lemma under which Theorem 2.3 holds with β^3 in I_2 replaced by β . On the set of parameters complementary to the set for which the singular continuous spectrum occurs we have dense pure point spectrum. The spectral transition is, consequently, sharp.

Our main result, the combination of Theorem 3.2 and Lemma 4.1, is summarized as follows:

Theorem 2.4 *Let $J_{P,\phi}$ be as in Theorem 2.3. Let I be the interval (2.24) (for the diagonal case, replace $(1 - p^2)/p$ by v), $A = 2 \cos \pi \mathbb{Q}$ and set*

$$\begin{aligned} A_{sc} &= A \cap I \\ A_{pp} &= A \cap I^c \end{aligned} \tag{2.26}$$

where I^c means complementary of I in $[-2, 2]$. Then, for almost all ω with respect to uniform product measure on $\Lambda = \times_{j=1}^\infty \{-j, \dots, j\}$,

(a) *the spectrum restricted to the set $I \setminus A'_{sc}$, with $A'_{sc} = A_{sc} \cup A'$ and A' a set of Lebesgue measure zero related with the definition of essential support of μ , is purely singular continuous;*

(b) *the spectrum of $J_{P,\phi}$ is dense pure point when restricted to $I^c \setminus A_{pp}$ for almost every $\phi \in [0, \pi)$, where ϕ characterizes the boundary condition (2.4). As we have only excluded a countable set A_{pp} , the spectrum is purely p.p. in $[-2, 2] \setminus I$.*

3 Uniform distribution of Prüfer angles

In this section we prove that the sequence of Prüfer angles $(\theta_k^\omega)_{k \geq 1}$ is u.d. mod π for almost every ω and for all $\varphi \in [0, \pi]$ such that φ/π is an irrational number. For this purpose, we make use of the following slight modification of the (optimal) metric extension of Weyl's criterion for uniform distribution by Davenport, Erdős and LeVeque [DEL] (see also Theorem 4.2 of [KN]).

Theorem 3.1 *Let $(u_n(x))_{n \geq 1}$ be a sequence of real-valued random variables defined in a probability space $(\Omega, \mathcal{B}, \mu)$. For integers $h \neq 0$, $N \geq 1$ and $A \subset \Omega$, we set*

$$S_h(N, x) = \frac{1}{N} \sum_{n=1}^N e^{2ihu_n(x)} \quad (3.1)$$

and

$$I_h(N, A) = \int_A |S_h(N, x)|^2 d\mu(x). \quad (3.2)$$

If the series $\sum_{N=1}^{\infty} I_h(N, A)/N$ converges for each $h \neq 0$, then the sequence $(u_n(x))$ is u.d. mod π for almost all $x \in A$ with respect to μ .

The random variables $\{\omega_j, j \geq 1\}$, defined on probability space $(\Omega, \mathcal{B}, \mu)$, are statistically independent and uniformly distributed in $\Lambda_j = \{-j, \dots, j\}$. Let $\Lambda = \times_{j=1}^{\infty} \Lambda_j$ be the configuration space. The measure μ induces on the mesurable space (Λ, \mathcal{F}) , where \mathcal{F} is the natural product σ -algebra, a product measure

$$\nu(B) = \prod_{j=1}^{\infty} \nu_j(B_j) = \mu(\omega^{-1}(B))$$

where $\omega^{-1}(B) = \bigcap_{j=1}^{\infty} \omega_j^{-1}(B_j)$, defined for all cylinder sets $B = \times_{j=1}^{\infty} B_j \subset \Lambda$ with ν_j the uniform measure in Λ_j : $\nu_j(k) = 1/(2j+1)$ for any $k \in \{-j, \dots, j\}$. We denote the sequence of Prüfer angles, defined by the recursive relations (2.17), either by $(\theta_k(x))_{k \geq 1}$ or by $(\theta_k^{\omega})_{k \geq 1}$, depending on whether the probability space $(\Omega, \mathcal{B}, \mu)$ or $(\Lambda, \mathcal{F}, \nu)$ is referred.

The main result of this section is as follows.

Theorem 3.2 *The sequence of Prüfer angles $(\theta_k(x))_{k \geq 1}$ ($(\theta_k^{\omega})_{k \geq 1}$) is u.d. mod π for all $\varphi/\pi \in [0, 1] \setminus \mathbb{Q}$ and all $x \in \Omega$ ($\omega \in \Lambda$) apart from a set with μ (ν) measure 0.*

Proof. According to Theorem 3.1, we must show that the series $\sum_{N=1}^{\infty} I_h(N, \Omega)/N$ converges, $I_h(N, \Omega)$ defined by (3.2). It is, nevertheless, sufficient to show that the series converges absolutely. Thus, by (2.17) and (3.1),

$$\begin{aligned} I_h(N, \Omega) &= \int_{\Omega} |S_h(N, x)|^2 d\mu(x) \\ &\leq \frac{1}{N} + \frac{2}{N^2} \sum_{1 \leq m < n \leq N} \left| \int_{\Omega} e^{2ih(\theta_m(x) - \theta_n(x))} d\mu(x) \right|. \end{aligned} \quad (3.3)$$

Since θ_m^ω with $m < n$ and $\tilde{\theta}_n^\omega$, given by $\theta_n^\omega = g(\theta_{n-1}^\omega, \varphi) - (\beta^n + \omega_n - \omega_{n-1})\varphi \equiv \tilde{\theta}_n^\omega - \omega_n\varphi$, are statistically independent of ω_n , we have

$$\begin{aligned} \left| \int_{\Omega} e^{2ih(\theta_m(x) - \theta_n(x))} d\mu(x) \right| &= \left| \int_{\Lambda} e^{2ih(\theta_m^\omega - \tilde{\theta}_n^\omega)} d\nu(\omega) \right| \left| \int_{\Lambda_n} e^{2ih\omega_n\varphi} d\nu_n(\omega_n) \right| \\ &\leq \left| \int_{\Lambda_n} e^{2ih\omega_n\varphi} d\nu_n(\omega_n) \right|. \end{aligned} \quad (3.4)$$

The r.h.s. of (3.4) is the characteristic function of ν_n at $2h\varphi$ and

$$\begin{aligned} \left| \frac{1}{2n+1} \sum_{k=-n}^n e^{2ihk\varphi} \right| &= \left| \frac{1}{2n+1} \frac{\sin(2n+1)h\varphi}{\sin h\varphi} \right| \\ &\leq \frac{1}{(2n+1)|\sin(h\varphi)|} < \infty, \end{aligned} \quad (3.5)$$

holds except if φ is a rational multiple of π in $[0, \pi]$, justifying the restriction stated in the theorem.

Together, equations (3.3), (3.4) and (3.5) yields

$$I_h(N, \Omega) \leq \frac{1}{N} + \frac{2}{|\sin h\varphi|N^2} \sum_{1 \leq m < n \leq N} \frac{1}{2n+1} < \frac{1}{N} \left(1 + \frac{1}{|\sin h\varphi|} \right),$$

which implies that $\sum_{N=1}^{\infty} I_h(N, \Omega)/N$ is finite for each $h \neq 0$, concluding the proof of Theorem 3.2. □

Remark 3.3 *If φ/π is any rational number, there exists a $h \in \mathbb{Z}$, $h \neq 0$, such that $h\varphi/\pi = m \in \mathbb{Z}$ and we have*

$$\lim_{\varphi' \rightarrow \varphi} \frac{\sin(2n+1)h\varphi'}{\sin h\varphi'} = 2n+1,$$

which implies that $I_h(N, \Omega) \leq O(1)$ and, consequently, the estimate of $\sum_{N=1}^{\infty} I_h(N, \Omega)/N$ employed on the proof of Theorem 3.2 diverges.

Remark 3.4 *The uniform assumption on ν_j is not necessary for our analysis, but it is assumed for simplicity. To assure that the Prüfer angles $(\theta_k^\omega)_{k \geq 1}$ are u.d. mod π it is sufficient to define the random variables ω_j supported in a interval $\Lambda_j \equiv [-j^\varepsilon, j^\varepsilon] \cap \mathbb{Z}$, ε any positive real number. By the proof of Theorem 3.2 it is clear that in this case we would have*

$$\sum_{N=1}^{\infty} \frac{I_h(N, \Omega)}{N} \leq \sum_{N=1}^{\infty} \frac{O(1)}{N^{1+\varepsilon}} < \infty,$$

which, by Theorem 3.1, proves our assertion.

Remark 3.5 *The proof of Theorem 3.2 independ on the sparseness condition (2.3). Although we are free to choose any function, Theorems 2.3 and 2.4 require that*

$$a_n^\omega - a_{n-1}^\omega \geq 2 \quad (3.6)$$

holds for all $n > 1$ and $\omega \in \Lambda$.

4 Improved Criterion for Pure Point Spectrum

The criterion (2.20) for pure point spectrum is not optimal for sparse Jacobi matrices. To obtain an interval stated in Theorem 2.4 we use instead the following

Lemma 4.1 *Let $J_{P,\phi}$ be as in Theorem 2.3 and let $t_n = \|T(a_n + 1; \lambda)\|$ denote the spectral norm of the associate transfer matrix $T(k; \lambda)$ at the point $k = a_n + 1$ for some $\lambda \in [-2, 2]$. If*

$$\sum_{n=1}^{\infty} \beta^n t_n^{-2} < \infty \quad (4.1)$$

and

$$\sum_{n=1}^{\infty} \beta^n t_n^2 \left(\sum_{m=n}^{\infty} t_m^{-2} \right)^2 < \infty \quad (4.2)$$

are verified, then the eigenvalue equation $J_{P,\phi}u = \lambda u$ has an $l_2(\mathbb{Z}_+)$ solution.

Proof. We begin by adapting Theorem 8.1 of [LS] for the model in consideration.

By (2.6) and (2.2),

$$\|T(k, k-1; \lambda)\|^2 \leq \|T(k, k-1; \lambda)\|_E^2 \leq 1 + \frac{1 + \lambda^2}{p^2} < \infty$$

if $p \in (0, 1)$, where $\|\cdot\|_E$ is the Euclidean matrix norm, for $k, k-1 \in \mathcal{A}$, otherwise $T_j(k, k-1; \lambda)$ is similar to a clockwise rotation $R(\varphi)$ by $\varphi = (1/2) \arccos \lambda$ (see equation (2.10)). We write

$$T(a_n + 1; \lambda) = A_n(\lambda) \cdots A_1(\lambda)$$

where, for each $m \geq 2$

$$A_m(\lambda) = T(a_m + 1, a_m; \lambda) \cdots T(a_{m-1} + 2, a_{m-1} + 1; \lambda) = T_- T_+ T_0^{\beta^m - 2}$$

by (2.9). Denoting by $s_n = \|A_n(\lambda)\|$ the spectral norm of $A_n(\lambda)$, we have

$$s_n \leq C \left(1 + \frac{1 + \lambda^2}{p^2} \right) \equiv B \quad (4.3)$$

$C = (1 + |\cos \varphi_j|)/(1 - |\cos \varphi_j|)$, uniformly in n . As a consequence,

$$\sum_{n=1}^{\infty} \frac{s_{n+1}^2}{t_n^2} < \infty \quad (4.4)$$

verifies the assumption of Theorem 8.1 of [LS] and provides the existence of a subordinate solution v associated to λ . Since the sparseness parameter satisfies $\beta \geq 2$, (4.1) is stronger than (4.4) and that assumption is already verified under the hypotheses of Lemma 4.1.

The transfer matrices $T_0, T_{+-} := T_+ T_-$ given by (2.9) are, together with $T(a_n + 1; \lambda)$ and $T^*(a_n + 1; \lambda)$, 2×2 unimodular real matrices. Hence the product $T^*(a_n + 1; \lambda)T(a_n + 1; \lambda)$ is a 2×2 unimodular symmetric real matrix whose eigenvalues are t_n^2 and t_n^{-2} , and corresponding (normalized) eigenvectors \mathbf{v}_n^+ and \mathbf{v}_n^- are orthogonal: $(\mathbf{v}_n^+, \mathbf{v}_n^-) = 0$. We write $\mathbf{v}_\phi = \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}$ and define ϕ_n by

$$\mathbf{v}_{\phi_n} = \mathbf{v}_n^- . \quad (4.5)$$

Clearly, $\mathbf{v}_n^+ = \mathbf{v}_{\phi_n + \pi/2}$ and by the spectral theorem, we have

$$\begin{aligned} \|T(a_n + 1; \lambda)\mathbf{v}_\phi\|^2 &= (\mathbf{v}_\phi, T^*(a_n + 1; \lambda)T(a_n + 1; \lambda)\mathbf{v}_\phi) \\ &= t_n^2 |(\mathbf{v}_\phi, \mathbf{v}_n^+)|^2 + t_n^{-2} |(\mathbf{v}_\phi, \mathbf{v}_n^-)|^2 \\ &= t_n^2 \sin^2(\phi - \phi_n) + t_n^{-2} \cos^2(\phi - \phi_n) . \end{aligned} \quad (4.6)$$

$u = (u_k)_{k \geq 0}$, given by the second component $u_k = (T(k; \lambda)\mathbf{v}_\phi)_2$, solves the eigenvalue equation $J_{P, \phi} u = \lambda u$ with boundary condition (2.4).

Using properties of matrix norm together with (4.6) for $n + 1$ and definition (4.5), it can be shown (see proof of Theorem 8.1 of [LS])

$$|\phi_n - \phi_{n+1}| \leq \frac{\pi}{2} \frac{s_{n+1}^2}{t_n^2} .$$

Condition (4.4) implies that the sequence $(\phi_n)_{n \geq 1}$ has a limit $\phi^* = \lim_{n \rightarrow \infty} \phi_n$. Hence, equation (4.6) and the telescope estimate

$$|\phi_n - \phi^*| \leq \sum_{m=n}^{\infty} |\phi_m - \phi_{m+1}| \leq \frac{\pi}{2} \sum_{m=n}^{\infty} \frac{s_{m+1}^2}{t_m^2}$$

yields

$$\begin{aligned} \|T(a_n + 1; \lambda)\mathbf{v}_{\phi^*}\|^2 &\leq t_n^2 (\phi^* - \phi_n)^2 + t_n^{-2} \\ &\leq \frac{\pi}{2} B^4 t_n^2 \left(\sum_{m=n}^{\infty} t_m^{-2} \right)^2 + t_n^{-2} \end{aligned} \quad (4.7)$$

Note that, by definition (2.7) of transfer matrix, $v_k = (T(k; \lambda) \mathbf{v}_{\phi^*})_2$ is a strongly subordinate solution in the sense that, for $u_k \equiv (T(k; \lambda) \mathbf{v}_{\phi^* + \pi/2})_2$, we have

$$\lim_{k \rightarrow \infty} \frac{|v_k^2 + v_{k+1}^2|}{|u_k^2 + u_{k+1}^2|} = 0 ,$$

since $\|T(a_n + 1; \lambda) \mathbf{v}_{\phi^* + \pi/2}\|^2 \geq t_n^2/2$ is satisfied for sufficiently large n and, for some n such that $a_n + 1 \leq k < a_{n+1}$, $\|T(k; \lambda) \mathbf{v}_{\phi^*}\| \leq B \|T(a_n + 1; \lambda) \mathbf{v}_{\phi^*}\|$ holds.

To conclude the proof, it remains to show that the subordinate solution is also $l_2(\mathbb{Z}_+)$ under the hypotheses (4.1) and (4.2). By the equivalence of norms it is enough to show it for the U -norm $\|\cdot\|_U := \|U \cdot\|$, where U is given by (2.11) with $\lambda = 2 \cos \varphi$. For any $k \in \mathbb{N}$, we pick n such that $a_n + 1 \leq k < a_{n+1}$ holds. We have from (2.14)

$$\|T(k; \lambda) \mathbf{v}_{\phi^*}\|_U^2 = \|T(a_n + 1; \lambda) \mathbf{v}_{\phi^*}\|_U^2 \leq (1 + |\cos \varphi|) \|T(a_n + 1; \lambda) \mathbf{v}_{\phi^*}\|^2$$

which, by the sparseness condition (2.3) together with (4.7), yields

$$\sum_{k=1}^{\infty} \|T(k; \lambda) \mathbf{v}_{\phi^*}\|_U^2 \leq B' \sum_{n=1}^{\infty} \beta^n t_n^2 \left(\sum_{m=n}^{\infty} t_m^{-2} \right)^2 + B'' \sum_{n=1}^{\infty} \beta^n t_n^{-2}$$

for some constants B' and B'' , concluding the proof. \square

Completion of the proof of Theorem 2.4. By (2.15)-(2.17), (2.21) and Theorem 3.2, we have for all $\varphi/\pi \in [0, 1] \setminus \mathbb{Q}$ and a.e. ω (see the analogues of (2.16) and (2.17) for the diagonal case in [Z]),

$$\frac{1}{p^2} C_N^{-1/N} \exp \left(\int_0^\pi f(\theta) d\theta \right) \leq (R_N^2)^{1/N} \leq \frac{1}{p^2} C_N^{1/N} \exp \left(\int_0^\pi f(\theta) d\theta \right) \quad (4.8a)$$

where, by Koksma's inequalities [K] (see also Theorem 5.1 in Chap. 2 of [KN]),

$$C_N \equiv \exp \left(-\frac{N D_N^*}{\pi} \int_0^\pi |f'(\theta)| d\theta \right) \quad (4.8b)$$

with D_N^* the discrepancy of the sequence $(\theta_k^\omega)_{k \geq 1}$ which tends to zero for u.d. sequences by a theorem of Weyl [W]) (see also Corolary 1.1 in Chap. 2 of [KN] or [DT]). This implies, by (4.8b):

$$\lim_{N \rightarrow \infty} C_N^{1/N} = 1 . \quad (4.8c)$$

By (2.14), (2.22), (4.8a) and the methods of [LS] (see [MWGA] pp 776–778 for a complete derivation)

$$\tilde{C}_n^{-1} r^n \leq t_n^2 \leq \tilde{C}_n r^n \quad (4.8d)$$

holds for every n , with r given by (2.23) and \tilde{C}_n satisfying (4.8c). By (4.8d), (4.1) and (4.2) are simultaneously satisfied provided $\lambda \in [-2, 2] \setminus 2 \cos \pi \mathbb{Q}$ and

$$\frac{\beta}{r} = \beta \left(1 + \frac{(1 - p^2)^2}{p^2(4 - \lambda^2)} \right)^{-1} < 1 ,$$

which is the condition complementary to the one defining in (2.24). Analogous results holds for the diagonal case, and this completes the proof of Theorem 2.4.

□

5 Sub and super-exponential sparseness

This section is devoted to the characterization of the spectral measure μ when the sparseness of the perturbations (2.2) is sub or super-exponential.

Our random set $\mathcal{A} = \{a_n^\omega\}_{n \geq 1}$ of natural numbers $a_n^\omega = a_n + \omega_n$ is now admissible if

$$a_n - a_{n-1} = \lceil e^{cn^\gamma} \rceil \quad (5.1)$$

holds for every $n \geq 1$, with $c, \gamma > 0$ and ω_n , $n \geq 1$, independent random variables, uniform in Λ_n . Here $[z]$ denotes the integer part of a real number z . The sequence defined by (5.1) increases sub or super-exponentially fast depending on whether $\gamma < 1$ or $\gamma > 1$, and coincides with the one previously defined if $\gamma = 1$ and $c = \ln \beta$. Other sparseness conditions with different growth rate may be dealt using the same methods employed in this section but the family chosen is already wide enough for our purposes.

Let the Jacobi matrix $J_{P,\phi}$ be defined by equations (2.1), (2.2) and (2.4) with the condition (2.3) replaced by (5.1). We shall also allow p in equation (2.2) vary in some cases. As the proof of Theorem 2.2 still holds in all these situations (see proof of Theorem 2.1 of [MWGA]), the essential spectrum of $J_{P,\phi}$ remain the same as before: $\sigma_{\text{ess}}(J_{P,\phi}) = [-2, 2]$.

5.1 $\gamma < 1$ case

We begin with the following

Theorem 5.1 *Let $J_{P,\phi}$ be as in Theorem 2.3 with the sparseness condition (2.3) replaced by (5.1) with $0 < \gamma < 1$ and $c > 0$. Then*

$$\sigma_{\text{ess}}(J_{P,\phi}) = \sigma_{pp}(J_{P,\phi}) = [-2, 2] \quad (5.2)$$

holds for almost every boundary condition $\phi \in [0, \pi)$ and almost all $\omega \in \Lambda = \times_{n=1}^\infty \{-j, \dots, j\}$ with respect to the product measure ν .

Proof. This theorem is an extension of Theorem 2.3 (b) for the case of sub-exponential sparseness. According to Theorem 1.7 from [LS], we need to show that condition (2.20) is satisfied for every $\lambda \in [-2, 2]$ and almost every boundary condition $\phi \in [0, \pi)$.² Theorem 2.2 then completes the proof.

²It is not necessary to apply the improved version Lemma 4.1.

If n is such that $a_N^\omega \leq n < a_{N+1}^\omega$ is satisfied for some $N \in \mathbb{N}$, by equations (4.19)-(4.23) of [MWGA], we have that

$$\sum_{k=n}^{\infty} \|T(k; \lambda)\|^{-2} \leq C^2 \sum_{l=N}^{\infty} (e^{cl^\gamma} + 2l) C_l r^{-l} \equiv S_N \quad (5.3)$$

with $C = \sqrt{(1 + |\cos \varphi|)/(1 - |\cos \varphi|)}$, holds for $\lambda \in 2 \cos([0, \pi] \setminus A')$, A' a fixed set of Lebesgue measure zero, and for all $\omega \in \Lambda$ apart a set of ν -measure zero. Note that Theorem 3.2 assures that $C_l^{-1} r^l \leq R_l^2 \leq C_l r^l$ holds with r defined by (2.23) and $C_l^{1/l} \searrow 1$ as $l \rightarrow \infty$.

Using (2.20) together with (5.3) give

$$N_{\text{pp}} \leq C' \sum_{N=1}^{\infty} (e^{cN^\gamma} + 2N) C_N^+ r^N S_N^2 \leq C'' \sum_{N=1}^{\infty} (e^{cN^\gamma} + 2N)^3 (C_N^+)^3 r^{-N}$$

for some finite constants C' and C'' . Hence, N_{pp} converges for all γ , $0 < \gamma < 1$, and for every λ in the set

$$B = [-2, 2] \setminus A_1, \quad (5.4)$$

A_1 a set of Lebesgue measure zero, the eigenvalue equation $J_{P,\phi} u = \lambda u$ has an $l_2(\mathbb{Z}_+)$ solution. Finally, since (2.5) remains true for every boundary condition $\phi \in [0, \pi)$, Proposition 4.3 of [MWGA] implies that $J_{P,\phi}$ has only dense pure point spectrum in $[-2, 2]$ for almost every $\phi \in [0, \pi)$. Thus, we have proved (5.2) and, consequently, the theorem. \square

The pure point spectrum in Theorem 5.1 holds for any perturbation $0 < p < 1$ (see equation (2.2)). A less singular spectrum can be obtained if we let $p_{a_k^\omega}$ tend to 1 as a_k^ω tends to ∞ .

5.2 $\gamma > 1$ cases

Let us consider now the Jacobi matrix $J_{P,\phi}$ subjected to the super-exponential sparseness condition (5.1) with $\gamma > 1$. The spectral measure of $J_{P,\phi}$ in this situation is given by Theorem 1.4(2) of [Z].

Theorem 5.2 *Let $J_{P,\phi}$ be defined as in Theorem 5.1 with $\gamma > 1$ and $c > 0$. Then the spectral measure ρ of $J_{P,\phi}$ is purely singular continuous and its Hausdorff dimension is 1 everywhere in $(-2, 2)$ for almost all $\omega \in \Lambda = \times_{n=1}^{\infty} \{-j, \dots, j\}$.*

Proof. Together with the simple estimate

$$\gamma(n+1)^{\gamma-1} \geq (n+1)^\gamma - n^\gamma = \int_n^{n+1} \gamma x^{\gamma-1} dx \geq \gamma n^{\gamma-1}, \quad (5.5)$$

equations (2.2) and (5.1), yields

$$\frac{a_n^\omega}{a_{n+1}^\omega} \leq \frac{e^{cn^\gamma} + 2n}{e^{c(n+1)^\gamma} - 2n} = \frac{1 + 2ne^{-cn^\gamma}}{e^{c\gamma n^{\gamma-1}} - 2ne^{-cn^\gamma}} \longrightarrow 0$$

as n tends to infinity, for all $\omega \in \Lambda$, $\gamma > 1$ and $c > 0$. Thus, by Theorem 1.4(2) of [Z], the Hausdorff dimension of ρ is 1.

□

Remark 5.3 *The techniques developed in [MWGA] can also be used to prove that ρ has Hausdorff dimension 1 (see [CMW1]).*

The sparse perturbation may be chosen so that the spectral measure, despite of being singular continuous, has Hausdorff dimension 0. For this, let the sequence $P = (p_n)_{n \geq 0}$ vary depending on the $a_k^\omega \in \mathcal{A}$:

$$p_n = \begin{cases} q_k & \text{if } n = a_k^\omega \in \mathcal{A}, \\ 1 & \text{if otherwise} \end{cases} \quad (5.6)$$

where q_k goes to zero, as $k \rightarrow \infty$, super-exponentially fast:

$$e^{cn^\gamma - c_1 n^\delta} \leq \prod_{k=0}^n q_k^{-2} \leq e^{cn^\gamma - c_2 n^\delta} \quad (5.7)$$

holds for some $c_1 \geq c_2 > 0$ and $\gamma > \delta > 1$ such that $\delta > \gamma - 1$ holds. In this case, f defined by (2.16) is no longer the same function in (2.15) and, for each $k = 1, 2, \dots$, we have

$$\begin{aligned} f_k(\theta) &= \ln \left(q_k^4 \cos^2 \theta + (\sin \theta + (1 - q_k^2) \cot \varphi \cos \theta)^2 \right) \\ &\equiv \ln (a_k + b_k \cos 2\theta + c_k \sin 2\theta) \end{aligned}$$

where

$$\begin{aligned} 2a_k &= (1 - q_k^2)^2 \cot^2 \varphi + 1 + q_k^4 \\ 2b_k &= (1 - q_k^2)^2 \cot^2 \varphi - 1 + q_k^4 \\ c_k &= (1 - q_k^2) \cot \varphi \end{aligned}$$

satisfies

$$a_k^2 = b_k^2 + c_k^2 + q_k^4.$$

An explicit calculation, yields

$$a_k - \sqrt{a_k^2 - q_k^4} \leq a_k + b_k \cos 2\theta + c_k \sin 2\theta \leq a_k + \sqrt{a_k^2 - q_k^4} = \sin^{-2} \varphi (1 + O(q_k^2)) \quad (5.8)$$

which can be used to draw the following conclusion

Theorem 5.4 *Let $J_{P,\phi}$ be defined by (2.1), (5.6), (5.1) and (2.4) with $\gamma > 1$ and $c > 0$. Then, for almost every boundary condition $\phi \in [0, \pi)$ and all $\omega \in \Lambda = \times_{n=1}^\infty \{-j, \dots, j\}$, the spectral measure ρ of $J_{P,\phi}$ restricted to $(-2, 2)$ is purely singular continuous and has 0 Hausdorff dimension.*

Proof. Let us begin proving that the essential spectrum of $J_{P,\phi}$ is entirely singular continuous. According to Theorem 1.6 from [LS], it is sufficient to show that (2.19) holds for every $\lambda \in [-2, 2]$ apart a set of zero Lebesgue measure. From (2.14), (2.15), (5.7) and (5.8),

$$\sum_{n=0}^{\infty} \|T(n; \lambda)\|^{-2} \geq \tilde{C}^{-2} \sum_{n=0}^{\infty} e^{cn\gamma} \prod_{k=0}^n q_k^2 \sin^{2n} \varphi \geq \tilde{C}^{-2} \sum_{n=0}^{\infty} e^{c_2 n^\delta} \sin^{2n} \varphi \quad (5.9)$$

diverges for $\varphi \neq 0, \pi$, which establishes the claim.

Now we apply Corollaries 4.2 and 4.5 of [JL]. Suppose that for some $\alpha \in [0, 1)$ and every λ in some Borel set A ,

$$\limsup_{l \rightarrow \infty} \frac{1}{l^{2-\alpha}} \sum_{n=0}^l \|T(n; \lambda)\|^2 < \infty. \quad (5.10)$$

Then the restriction $\rho(A \cap \cdot)$ of the spectral measure $\mu = d\rho$ is α -continuous. Suppose, on the other hand, that

$$\liminf_{l \rightarrow \infty} \frac{\|u_{\text{sub}}\|_l^2}{l^\alpha} = 0 \quad (5.11)$$

for every λ in some Borel set A , where u_{sub} is a subordinate solution of $J_{P,\phi} u = \lambda u$. Then the restriction $\rho(A \cap \cdot)$ is α -singular.

By a calculation analogous to (5.9), for $a_m < l \leq a_{m+1}$

$$\begin{aligned} \sum_{n=0}^l \|T(n; \lambda)\|^2 &\leq C^2 \sum_{k=1}^m e^{ck\gamma} \prod_{j=0}^k q_j^{-2} \sin^{-2k} \varphi + C^2 (l - a_m) \prod_{j=0}^m q_j^{-2} \sin^{-2m} \varphi \\ &\leq C^2 \sum_{k=1}^m e^{2ck\gamma} e^{-c_2 k^\delta} \sin^{-2k} \varphi + C^2 (l - a_m) e^{cm\gamma} e^{-c_2 m^\delta} \sin^{-2m} \varphi \\ &\leq C' \left(\sum_{k=0}^m e^{2ck\gamma} + (l - a_m) e^{cm\gamma} \right) \leq C'' l^2 \end{aligned}$$

implies that ρ is α -continuous only for $\alpha = 0$. Since, by Lemma 2.1 of [Z], a subordinate solution of $J_{P,\phi} u = \lambda u$, $u_{\text{sub}}(n)$, decays as fast as the transfer matrix $\|T(n; \lambda)\|$ grows, we have

$$\|u_{\text{sub}}\|_l^2 \leq C^{-2} \sum_{n=0}^l \|T(n; \lambda)\|^{-2} \leq C^{-2} \sum_{k=0}^{m+1} e^{c_1 k^\delta} \sin^{2k} \varphi \leq C' e^{c_1 m^\delta} \quad (5.12)$$

for some finite constant C' . Since $l^\alpha \geq e^{cam\gamma}$ and $\gamma > \delta$, the inferior limit (5.11) goes to 0 for any $\alpha > 0$ and the spectral measure ρ is α -singular in the whole interval $(-2, 2)$ for any α concluding the proof of the theorem. □

Remark 5.5 Note that $q_k^{-2} = e^{c\gamma k^{\gamma-1} - c_1 \delta k^{\delta-1}}$, with $\gamma > \delta > 1$, $\delta > \gamma - 1$ and c_1 sufficiently large, satisfies (5.7), in view of

$$n^\kappa = \kappa \int_0^n x^{\kappa-1} dx < \sum_{k=0}^n \kappa k^{\kappa-1} < \kappa \int_0^{n+1} x^{\kappa-1} dx = (n+1)^\kappa$$

for $\kappa = \gamma$ and δ together with inequality (5.5).

Remark 5.6 It follows from equations (5.9) and (5.12) with $\delta = 1$ that the spectral measure restricted to $\{\lambda \in [-2, 2] : (1 - \lambda^2/4) e^{c_2} > 1\}$ is singular continuous with 0 Hausdorff dimension and pure point when restricted to $\{\lambda \in [-2, 2] : (1 - \lambda^2/4) e^{c_1} < 1\}$.

6 Conclusions and outlook

The transition depicted in Theorem 2.4 may be considered as an Anderson-like transition in one dimension, along the lines of the (multidimensional) program laid out by Molchanov [Mo]. Of particular relevance in this context is the important fact, shown by [DJLS] that Anderson localization is unstable under rank one perturbations, i.e., a rank one perturbation may change the p.p. spectrum into s.c. The same authors point out, however, that this s.c. spectrum must have zero Hausdorff dimension. Since the (local) Hausdorff dimension of the s.c. spectrum in Theorem 2.4 is (a.e.) nonzero, we conclude that the transition is robust. Also in this connection, since we are close to the difference Laplacian (in this regime of strong sparsity), what is really surprising in Theorem 2.4 is the existence of the pure point spectrum. One interesting possibility of pursuing this program is to analyse the Kronecker sum of two or more copies of $J_{P,\phi}$ [CMW2]. Following an idea of Malozemov-Molchanov [MaMo], the continuous part of the spectral measure may turn out to be absolutely continuous: this has been established, however, only for some classes of potentials with superexponential sparsity ([S],[KR], [CMW2]), for which, however, the spectrum is purely s.c. (see Theorem 5.2), and proving that the continuous part of models with mixed spectrum in two or more dimensions, such as the Kronecker sum of two or more copies of the present or similar models, becomes absolutely continuous, presents a challenging open problem.

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